

# Approximative characteristics of classes of functions $S_{p,\theta}^\Omega B(\mathbb{R}^d)$ with a given majorant of mixed modulus of smoothness

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## Abstract

We obtain order estimates of approximation of functions from the classes  $S_{p,\theta}^\Omega B(\mathbb{R}^d) \subset L_q(\mathbb{R}^d)$  by entire functions of exponential type with supports of their Fourier transforms in sets generated by the level surfaces of a function  $\Omega$ .

In the paper we continue to study the approximative characteristics of the functions from Nikolskii–Besov type classes  $S_{p,\theta}^\Omega B(\mathbb{R}^d)$  in the space  $L_q(\mathbb{R}^d)$  (see [1]). We established the order estimates of the best approximation of functions from these classes by entire functions of exponential type whose Fourier transforms are supported in sets generated by the level surfaces of a function  $\Omega$  when  $1 < p, q < \infty$ .

**1. Definition of classes of functions and approximative characteristics.** Let  $\mathbb{R}^d$  be the  $d$ -dimensional Euclidean space with the elements  $\mathbf{x} = (x_1, \dots, x_d)$ , and  $(\mathbf{x}, \mathbf{y}) = x_1 y_1 + \dots + x_d y_d$ . We will denote by  $L_q(\mathbb{R}^d)$ ,  $1 \leq q \leq \infty$ , the space of measurable functions on  $\mathbb{R}^d$  with the finite norm

$$\|f\|_{L_q} := \|f\|_q := \left( \int_{\mathbb{R}^d} |f(\mathbf{x})|^q d\mathbf{x} \right)^{\frac{1}{q}}, \quad 1 \leq q < \infty,$$

$$\|f\|_{L_\infty} := \|f\|_\infty := \operatorname{ess\,sup}_{\mathbf{x} \in \mathbb{R}^d} |f(\mathbf{x})|.$$

For  $f \in L_q(\mathbb{R}^d)$ , we will consider the difference of  $l$ -order,  $l \in \mathbb{N}$ , with respect to the variable  $x_j$  with step  $h_j$  which is defined as follows

$$\Delta_{h_j}^l f(\mathbf{x}) := \sum_{n=0}^l (-1)^{l-n} C_l^n f(x_1, \dots, x_{j-1}, x_j + nh_j, x_{j+1}, \dots, x_d).$$

Also, we note multiple mixed difference of order  $l$  of function  $f$  with vector step  $\mathbf{h} = (h_1, \dots, h_d)$  by

$$\Delta_{\mathbf{h}}^l f(\mathbf{x}) = \Delta_{h_d}^l (\Delta_{h_{d-1}}^l \dots (\Delta_{h_1}^l f(\mathbf{x}))).$$

The mixed modulus of smoothness of order  $l$  of function  $f \in L_q(\mathbb{R}^d)$  is defined according to the formula

$$\Omega_l(f, \mathbf{t})_q := \sup_{|\mathbf{h}| \leq \mathbf{t}} \|\Delta_{\mathbf{h}}^l f(\cdot)\|_q,$$

where  $|\mathbf{h}| = (|h_1|, \dots, |h_d|)$ . The type inequality  $\mathbf{a} \leq \mathbf{b}$  ( $\mathbf{a} > \mathbf{b}$ ) for vectors  $\mathbf{a} = (a_1, \dots, a_d)$  and  $\mathbf{b} = (b_1, \dots, b_d)$  here and further we understand coordinatewise, namely  $a_j \leq b_j$  ( $a_j > b_j$ ),  $j = \overline{1, d}$ . Also, we will use the notation  $\mathbf{t} \geq 0$ , if  $t_j \geq 0$ ,  $j = \overline{1, d}$ .

Let  $\Omega(\mathbf{t})$ ,  $\mathbf{t} = (t_1, \dots, t_d)$ , be a given function of mixed  $l$ th-order modulus of smoothness type, i.e., a function defined and continuous on  $\mathbb{R}_+^d$  which satisfies the following conditions:

- 1)  $\Omega(\mathbf{t}) > 0$ ,  $\mathbf{t} > 0$  and  $\Omega(\mathbf{t}) = 0$  if  $\prod_{j=1}^d t_j = 0$ ;
- 2)  $\Omega(\mathbf{t})$  is non-decreasing in each variable;
- 3)  $\Omega(m_1 t_1, \dots, m_d t_d) \leq \left( \prod_{j=1}^d m_j \right)^l \Omega(\mathbf{t})$ ,  $m_j \in \mathbb{N}$ ,  $j = \overline{1, d}$ .

We will denote a set of such functions  $\Omega$  by  $\Psi_l$ .

Additionally we will require that the function  $\Omega$  satisfies conditions  $(S^\alpha)$  and  $(S_l)$  which are called the Bari–Stechkin conditions [2]. These conditions are formulated as follows:

- a) We say that a function  $\varphi(\tau) \geq 0$  of one variable satisfies condition  $(S^\alpha)$ , if there is  $\alpha > 0$ , such that the function  $\varphi(\tau)/\tau^\alpha$  is almost increasing, i.e., there exists a constant  $C_1 > 0$  independently of  $\tau_1$  and  $\tau_2$  such that

$$\frac{\varphi(\tau_1)}{\tau_1^\alpha} \leq C_1 \frac{\varphi(\tau_2)}{\tau_2^\alpha}, \quad 0 < \tau_1 \leq \tau_2 \leq 1;$$

- b) We say that a function  $\varphi(\tau) \geq 0$  of one variable satisfies condition  $(S_l)$ , if there is  $\gamma$ ,  $0 < \gamma < l$ , such that the function  $\varphi(\tau)/\tau^{l-\gamma}$  is almost decreasing, i.e., there exists a constant  $C_2 > 0$  independently of  $\tau_1$  and  $\tau_2$  such that

$$\frac{\varphi(\tau_1)}{\tau_1^{l-\gamma}} \geq C_2 \frac{\varphi(\tau_2)}{\tau_2^{l-\gamma}}, \quad 0 < \tau_1 \leq \tau_2 \leq 1.$$

We will assume that  $\Omega$  satisfies conditions  $(S^\alpha)$  and  $(S_l)$  if  $\Omega$  satisfies these conditions for each variable  $t_j$  at fixed values of all other variables  $t_i$ ,  $i \neq j$ . In the case, when  $\Omega$  satisfies condition  $(S^\alpha)$ , we will say that  $\Omega$  belongs to the set  $S^\alpha$ , if  $\Omega$  satisfies condition  $(S_l)$ , then  $\Omega$  belongs to the set  $S_l$ . Asserting it (also for functions  $\omega$  of one variable), we use the notation  $\Omega \in \Phi_{\alpha, l}$ , ( $\omega \in \Phi_{\alpha, l}$ ),  $l \in \mathbb{N}$ , where the set  $\Phi_{\alpha, l}$  is determined by the relation  $\Phi_{\alpha, l} = \Psi_l \cap S^\alpha \cap S_l$ .

We will note that, for example, the functions

$$\Omega(\mathbf{t}) = \Omega(t_1, \dots, t_d) = \begin{cases} \prod_{j=1}^d \frac{t_j^{r_j}}{\left\{ \log \frac{1}{t_j} \right\}_+^{b_j}}, & \text{if } t_j > 0, j = \overline{1, d}; \\ 0, & \text{if } \prod_{j=1}^d t_j = 0, \end{cases}$$

where  $\{\log \tau\}_+ = \max\{1, \log_2 \tau\}$ ,  $r_j, b_j \in \mathbb{R}$ ,  $0 < r_j < l$ ,  $j = \overline{1, d}$ , belong to the set  $\Phi_{\alpha, l}$ .

Further, let  $e_d := \{1, 2, \dots, d\}$ ,  $d \in \mathbb{N}$ , and  $e := \{j_1, \dots, j_m\}$ ,  $m \leq d$ ,  $m \in \mathbb{N}$ ,  $1 \leq j_1 < j_2 < \dots < j_m \leq d$ ,  $\mathbf{t}^e = (t_{j_1}, \dots, t_{j_m})$ ,  $\bar{\mathbf{t}}^e := (\bar{t}_1, \dots, \bar{t}_d)$ , where

$$\bar{t}_i = \begin{cases} t_i, & i \in e, \\ 1, & i \in e_d \setminus e. \end{cases}$$

The spaces  $S_{p,\theta}^\Omega B(\mathbb{R}^d)$  for  $1 \leq p, \theta \leq \infty$  and  $\Omega \in \Psi_l$  are defined as follows (see., e.g., [1])

$$S_{p,\theta}^\Omega B(\mathbb{R}^d) := \left\{ f \in L_p(\mathbb{R}^d) : \|f\|_{S_{p,\theta}^\Omega B(\mathbb{R}^d)} < \infty \right\},$$

where

$$\|f\|_{S_{p,\theta}^\Omega B(\mathbb{R}^d)} := \|f\|_p + \sum_{\substack{e \subset e_d \\ e \neq \emptyset}} \left( \int_0^2 \dots \int_0^2 \left( \frac{\Omega_{l^e}(f, \mathbf{t}^e)_p}{\Omega(\bar{\mathbf{t}}^e)} \right)^\theta \prod_{j \in e} \frac{dt_j}{t_j} \right)^{\frac{1}{\theta}},$$

if  $1 \leq \theta < \infty$ , and

$$\|f\|_{S_{p,\infty}^\Omega B(\mathbb{R}^d)} := \|f\|_p + \sum_{\substack{e \subset e_d \\ e \neq \emptyset}} \sup_{\mathbf{t}^e > 0} \frac{\Omega_{l^e}(f, \mathbf{t}^e)_p}{\Omega(\bar{\mathbf{t}}^e)},$$

where

$$\Omega_{l^e}(f, \mathbf{t}^e)_q := \sup_{|\mathbf{h}^e| \leq \mathbf{t}^e} \|\Delta_{\mathbf{h}^e}^{l^e} f(\mathbf{x})\|_q, \quad \mathbf{h}^e := (h_{j_1}, \dots, h_{j_m}),$$

$$\Delta_{\mathbf{h}^e}^{l^e} f(\mathbf{x}) = \Delta_{h_{j_m}}^{l_{j_m}} (\Delta_{h_{j_{m-1}}}^{l_{j_{m-1}}} \dots (\Delta_{h_{j_1}}^{l_{j_1}} f(\dots, x_{j_1}, \dots, x_{j_m}, \dots))).$$

The function spaces  $S_{p,\theta}^\Omega B(\mathbb{R}^d)$  are generalizations of the known spaces  $S_{p,\theta}^r B(\mathbb{R}^d)$  [3], [4], that are determined at particular functions  $\Omega$ , namely  $\Omega(\mathbf{t}) = \mathbf{t}^r = t_1^{r_1} \dots t_d^{r_d}$ ,  $0 < r_j < l$ ,  $j = \overline{1, d}$ . We will remind that the spaces  $S_p^r H(\mathbb{R}^d) = S_{p,\infty}^r B(\mathbb{R}^d)$  have been first considered by S.M. Nikolskii [3], and the spaces  $S_{p,\theta}^r B(\mathbb{R}^d)$ ,  $1 \leq \theta < \infty$ , have been introduced by T.I. Amanov [4]. We will use the notation  $S_{p,\theta}^\Omega B$ ,  $S_{p,\theta}^r B$  and  $S_p^r H$  instead of  $S_{p,\theta}^\Omega B(\mathbb{R}^d)$ ,  $S_{p,\theta}^r B(\mathbb{R}^d)$  and  $S_p^r H(\mathbb{R}^d)$  respectively.

Research of Nikolskii–Besov classes with certain mixed smoothness  $S_{p,\theta}^r B(\mathbb{R}^d)$ , from the point of view of establishment of order estimates of some approximative characteristics were conducted, in particular, in the articles of Wang Heping and Sun Yongsheng [5], [6], Wang Heping [7], [8]. In the periodic case these classes of functions were investigated in the many articles, among the latter in particular, Wang Heping and Sun Yongsheng [9], Wang Heping [10], Song Zhanjie, Ye Peixin [11]. With the main results of research of Nikolskii–Besov classes with dominated mixed difference in the periodic case can be found in the monograph of V.N. Temkyakov [12], if  $\theta = \infty$  (for Nikolskii classes) and in the monograph of A.S. Romanyuk [13], if  $1 \leq \theta < \infty$  (for Besov classes). Currently there is considerable interest to study of different analogues of Nikolskii–Besov classes which are determined smooth parameter  $\Omega$ , for which executed some additional conditions: N. N. Pustovoitov [14]–[16], Wang Heping and Sun Yongsheng [17], Liqin Duan [18], Wang Heping, Sai Tang [19], Heping Wang, Kai Wang [20] and others.

In [1] there was proved the equivalent to rationing of linear spaces  $S_{p,\theta}^\Omega B(\mathbb{R}^d)$  indirectly through so-called decomposition representation of the elements of these spaces (see below Theorem A). Note that, decomposition representation and corresponding rationing for

Nikolskii–Besov spaces have been first obtained by S. N. Nikolskii and P. I. Lizorkin [21]. As it turned out, this decomposition norm of functions plays a pivotal role in the studies of different approximative characteristics of the function classes. Since the result from [1] using the quantities which are determined using the Fourier transforms of functions that defined on  $\mathbb{R}^d$ , then we give the corresponding definitions.

Let  $S = S(\mathbb{R}^d)$  be a Schwartz space of complex-valued, smooth, rapidly decreasing functions  $\varphi(\mathbf{x})$  on  $\mathbb{R}^d$  i.e., is infinitely differentiable and all its derivatives go to zero at infinity faster than any power of  $(x_1^2 + \dots + x_d^2)^{-\frac{1}{2}}$  (see, e.g., [22]). By  $S'$  we denote the space of linear continuous functionals on  $S$ . The elements of the space  $S'$  are generalized functions. If  $f \in S'$ , then  $\langle f, \varphi \rangle$  denotes the value of the functional  $f$  on the test function  $\varphi \in S$ .

The Fourier transform  $\mathfrak{F}\varphi : S \rightarrow S$  is defined by the formula

$$(\mathfrak{F}\varphi)(\boldsymbol{\lambda}) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \varphi(\mathbf{t}) e^{-i(\boldsymbol{\lambda}, \mathbf{t})} d\mathbf{t} \equiv \tilde{\varphi}(\boldsymbol{\lambda}).$$

The inverse Fourier transform is defined by the formula

$$(\mathfrak{F}^{-1}\varphi)(\mathbf{t}) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \varphi(\boldsymbol{\lambda}) e^{i(\boldsymbol{\lambda}, \mathbf{t})} d\boldsymbol{\lambda} \equiv \hat{\varphi}(\mathbf{t}).$$

The Fourier transform (the inverse Fourier transform, respectively) of generalized functions  $f \in S'$  is defined by the formula

$$\langle \mathfrak{F}f, \varphi \rangle = \langle f, \mathfrak{F}\varphi \rangle, \quad \langle \tilde{f}, \varphi \rangle = \langle f, \tilde{\varphi} \rangle, \quad \varphi \in S,$$

$$(\langle \mathfrak{F}^{-1}f, \varphi \rangle = \langle f, \mathfrak{F}^{-1}\varphi \rangle, \quad \langle \hat{f}, \varphi \rangle = \langle f, \hat{\varphi} \rangle, \quad \varphi \in S),$$

we will use the same notation for them.

The support of a generalized function  $f$  is the closure  $\overline{\mathfrak{N}}$  of the set  $\mathfrak{N} \subset \mathbb{R}^d$  with the property that for each function  $\varphi \in S$ , vanishing on  $\overline{\mathfrak{N}}$  we have  $\langle f, \varphi \rangle = 0$ . We will denote the support of a generalized function  $f$  by  $\text{supp } f$ . We say that a function  $f$  is concentrated on a set  $G$  if  $\text{supp } f \subseteq G$ .

Note that for  $1 \leq p \leq \infty$  the space  $L_p(\mathbb{R}^d)$  is naturally continuously embedded in  $S'$ . In this sense we identify functions from  $L_p(\mathbb{R}^d)$  with elements of  $S'$ .

Further, for each vector  $\mathbf{s} = (s_1, \dots, s_d)$ ,  $s_j \in \mathbb{Z}_+$ ,  $j = \overline{1, d}$ , we consider the set

$$Q_{2^{\mathbf{s}}}^* := Q^*(\mathbf{s}) := \left\{ \boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_d) : \eta(s_j) 2^{s_j-1} \leq |\lambda_j| < 2^{s_j}, \quad \lambda_j \in \mathbb{R}, \quad j = \overline{1, d} \right\},$$

where  $\eta(0) = 0$  and  $\eta(t) = 1$ ,  $t > 0$ .

Let  $\mathcal{A} \subset \mathbb{R}^d$  be a measurable set. Denote by  $\chi_{\mathcal{A}}$  the characteristic function of the set  $\mathcal{A}$ . For  $f \in L_p(\mathbb{R}^d)$ , we denote

$$\delta_{\mathbf{s}}^*(f, \mathbf{x}) = \mathfrak{F}^{-1}(\chi_{Q_{2^{\mathbf{s}}}^*} \cdot \mathfrak{F}f).$$

**Theorem A [1].** Let  $1 < p < \infty$  and  $\Omega \in \Phi_{\alpha,l}$ . A function  $f \in L_p(\mathbb{R}^d)$  belongs to the space  $S_{p,\theta}^\Omega B$ ,  $1 \leq \theta < \infty$ , if and only if

$$\left\{ \sum_{s \geq 0} (\Omega(2^{-s}))^{-\theta} \|\delta_s^*(f, \cdot)\|_p^\theta \right\}^{\frac{1}{\theta}} < \infty.$$

In this case,

$$\|f\|_{S_{p,\theta}^\Omega B} \asymp \left\{ \sum_{s \geq 0} (\Omega(2^{-s}))^{-\theta} \|\delta_s^*(f, \cdot)\|_p^\theta \right\}^{\frac{1}{\theta}}, \quad (1)$$

where  $\Omega(2^{-s}) = \Omega(2^{-s_1}, \dots, 2^{-s_d})$ .

A function  $f \in L_p(\mathbb{R}^d)$  belongs to the space  $S_{p,\infty}^\Omega B$ , if and only if

$$\sup_{s \geq 0} \frac{\|\delta_s^*(f, \cdot)\|_p}{\Omega(2^{-s})} < \infty.$$

In this case,

$$\|f\|_{S_{p,\infty}^\Omega B} \asymp \sup_{s \geq 0} \frac{\|\delta_s^*(f, \cdot)\|_p}{\Omega(2^{-s})}. \quad (2)$$

In what follows, the notation  $A \asymp B$  for positive quantities  $A$  and  $B$  means that there are positive constants  $C_3$  and  $C_4$ , that do not depend on an essential parameter in the values  $A$  and  $B$  (e.g.,  $C_3$  and  $C_4$  in the expressions (1) and (2) do not depend on the function  $f$ ) such that  $C_3 A \leq B \leq C_4 A$ . If only  $B \leq C_4 A$  ( $B \geq C_3 A$ ), we will write  $B \ll A$  ( $B \gg A$ ). All constants  $C_i$ ,  $i = 1, 2, \dots$ , in the present paper depend, possibly, only on the parameters contained in the definition of the function class, the metric in which we estimate the error of approximation, and the dimension of the space  $\mathbb{R}^d$ .

In what follows, the class  $S_{p,\theta}^\Omega B$  is understood as the set of functions  $f \in L_p(\mathbb{R}^d)$  with  $\|f\|_{S_{p,\theta}^\Omega B} \leq 1$ . For the classes  $S_{p,\theta}^\Omega B$  we use the same notation as for the spaces  $S_{p,\theta}^\Omega B$ .

Now we will define the approximative characteristics.

Let  $\mathcal{L} \subset \mathbb{Z}_+^d$  be a finite set. Put

$$Q(\mathcal{L}) = \bigcup_{s \in \mathcal{L}} Q^*(s)$$

and denote

$$G(Q(\mathcal{L})) = \left\{ f \in L_q(\mathbb{R}^d) : \text{supp } \mathfrak{F}f \subseteq Q(\mathcal{L}) \right\}.$$

We know that elements of the set  $G(Q(\mathcal{L}))$  are entire functions of exponential type.

For  $f \in L_q(\mathbb{R}^d)$ ,  $1 \leq q \leq \infty$ , consider the quantity

$$E(f, G(Q(\mathcal{L})))_q := E_{Q(\mathcal{L})}(f)_q := \inf_{g \in G(Q(\mathcal{L}))} \|f(\cdot) - g(\cdot)\|_q,$$

which is called the best approximation of  $f$  by entire functions from the set  $G(Q(\mathcal{L}))$ . If  $F \subset L_q(\mathbb{R}^d)$  is a function class, we put

$$E_{Q(\mathcal{L})}(F)_q = \sup_{f \in F} E_{Q(\mathcal{L})}(f)_q. \quad (3)$$

Further for  $f \in L_q(\mathbb{R}^d)$ ,  $1 \leq q \leq \infty$ , put

$$S_{Q(\mathcal{L})}f(\mathbf{x}) = S_{Q(\mathcal{L})}(f, \mathbf{x}) = \sum_{\mathbf{s} \in \mathcal{L}} \delta_{\mathbf{s}}^*(f, \mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^d,$$

and we define

$$\mathcal{E}_{Q(\mathcal{L})}(f)_q := \|f(\cdot) - S_{Q(\mathcal{L})}f(\cdot)\|_q, \quad \mathcal{E}_{Q(\mathcal{L})}(F)_q := \sup_{f \in F} \mathcal{E}_{Q(\mathcal{L})}(f)_q. \quad (4)$$

Our investigations of quantities (3) and (4) are held when  $F = S_{p,\theta}^\Omega B(\mathbb{R}^d)$  and the set  $\mathcal{L}$  in some way associated with function  $\Omega$ .

For each  $N \in \mathbb{N}$  put

$$\begin{aligned} \kappa(\Omega, N) &:= \kappa(N) := \left\{ \mathbf{s} = (s_1, \dots, s_d) \in \mathbb{Z}_+^d : \Omega(2^{-\mathbf{s}}) 2^{\|\mathbf{s}\|_1(\frac{1}{p}-\frac{1}{q})_+} \geq \frac{1}{N} \right\}, \\ Q(\kappa(N)) &:= Q(N) =: \bigcup_{\mathbf{s} \in \kappa(N)} Q^*(\mathbf{s}), \end{aligned}$$

where  $\|\mathbf{s}\|_1 := s_1 + \dots + s_d$ ,  $a_+ := \max\{0, a\}$ .

Note that the sets  $Q(N)$  are generated by the level sets of the function  $\Omega(\mathbf{t})\mathbf{t}^{-(\frac{1}{p}-\frac{1}{q})_+}$ ,  $\Omega(\mathbf{t}) \in \Phi_{\alpha,l}$ ,  $\alpha > \left(\frac{1}{p} - \frac{1}{q}\right)_+$ . If

$$\Omega(\mathbf{t}) = \Omega_1(\mathbf{t}) / \prod_{j=1}^d t_j^{-(\frac{1}{p}-\frac{1}{q})_+}$$

and  $\Omega_1(\mathbf{t}) = \prod_{j=1}^d t_j^{r_j}$ ,  $0 < r_j < l$ ,  $j = \overline{1, d}$ , we get the sets  $Q(N)$ , which called step hyperbolic crosses.

Note that the approximation of Nikol'skii–Besov classes of periodic functions of mixed smoothness by trigonometric polynomials with spectrum in step hyperbolic crosses and in the sets  $Q(N)$  were considered, in particular, in [15], [16], [23]–[31]. With classical results and methods, which are designed for approach in step hyperbolic cross can be found in the recent survey by Dinh Dung, Vladimir Temlyakov, Tino Ullrich [32].

For the formulation of auxiliary statements and proofs of our main results we will define some sets in  $\mathbb{Z}_+^d$ .

We put

$$\kappa^\perp(\Omega, N) := \kappa^\perp(N) := \left\{ \mathbf{s} = (s_1, \dots, s_d) \in \mathbb{Z}_+^d : \Omega(2^{-\mathbf{s}}) 2^{\|\mathbf{s}\|_1(\frac{1}{p}-\frac{1}{q})_+} < \frac{1}{N} \right\},$$

$$Q^\perp(\kappa^\perp(N)) := Q^\perp(N) := \bigcup_{\mathbf{s} \in \kappa^\perp(N)} Q^*(\mathbf{s}),$$

$$\Theta(N) := \kappa^\perp(N) \setminus \kappa^\perp(2^l N),$$

i.e.,

$$\Theta(N) = \left\{ \mathbf{s} = (s_1, \dots, s_d) \in \mathbb{Z}_+^d : \frac{1}{2^l N} \leq \Omega(2^{-\mathbf{s}}) 2^{\|\mathbf{s}\|_1(\frac{1}{p}-\frac{1}{q})_+} < \frac{1}{N} \right\} \quad (5)$$

In [15] has been shown that

$$|\Theta(N)| \asymp (\log_2 N)^{d-1}, \quad (6)$$

where  $|A|$  denotes the number of elements of a finite set  $A$ .

The following statements are true.

**Lemma A** [15]. *Let  $\Omega$  be a function of mixed modulus of smoothness of order  $l$  type, and let  $\Omega$  satisfies condition  $(S^\alpha)$ ,  $\alpha > 0$ . Then, for  $0 < \mu < \infty$  we have*

$$\sum_{s \in \kappa^\perp(N)} (\Omega(2^{-s}))^\mu \ll \sum_{s \in \Theta(N)} (\Omega(2^{-s}))^\mu. \quad (7)$$

**Lemma B** [15]. *Let  $\Omega$  be a function of mixed modulus of smoothness of order  $l$  type, and let  $\Omega$  satisfies condition  $(S^\alpha)$  with  $\alpha > \beta > 0$ . Then, for  $0 < \mu < \infty$  we have*

$$\sum_{s \in \kappa^\perp(N)} (\Omega(2^{-s}) 2^{\|s\|_1 \beta})^\mu \ll \sum_{s \in \Theta(N)} (\Omega(2^{-s}) 2^{\|s\|_1 \beta})^\mu. \quad (8)$$

As a consequence of (7), (8), (5) and (6) for  $0 < \mu < \infty$  we have

$$\begin{aligned} \sum_{s \in \kappa^\perp(N)} \left( \Omega(2^{-s}) 2^{\|s\|_1 \left(\frac{1}{p} - \frac{1}{q}\right)_+} \right)^\mu &\ll \sum_{s \in \Theta(N)} \left( \Omega(2^{-s}) 2^{\|s\|_1 \left(\frac{1}{p} - \frac{1}{q}\right)_+} \right)^\mu < \\ &< \left( \frac{1}{N} \right)^\mu \sum_{s \in \Theta(N)} 1 = \left( \frac{1}{N} \right)^\mu |\Theta(N)| \asymp \left( \frac{1}{N} \right)^\mu (\log_2 N)^{d-1}. \end{aligned} \quad (9)$$

We formulate an additional the following statement essentially used in our subsequent presentation.

**Theorem B** (Littlewood–Paley) (see, e.g., [33, 1.5.6], [34]). *Let  $1 < p < \infty$ . There exist positive numbers  $C_5, C_6$  such that for each function  $f \in L_p(\mathbb{R}^d)$  we have*

$$C_5 \|f\|_p \leq \left\| \left( \sum_{s \geq 0} |\delta_s^*(f, \cdot)|^2 \right)^{\frac{1}{2}} \right\|_p \leq C_6 \|f\|_p.$$

In what follows, we also use a lemma which is similar of the corresponding lemma in the periodic case [12, p. 25].

**Lemma C** [5]. *Let  $1 < p < q < \infty$  and  $f \in L_p(\mathbb{R}^d)$ . Then*

$$\|f\|_q \ll \left( \sum_{s \geq 0} \|\delta_s^*(f, \cdot)\|_p^q 2^{\|s\|_1 \left(\frac{1}{p} - \frac{1}{q}\right)q} \right)^{\frac{1}{q}}.$$

**2. Order estimates of the approximative characteristics of the classes  $S_{p,\theta}^\Omega B(\mathbb{R}^d)$  in the metric of space  $L_q(\mathbb{R}^d)$ ,  $1 < p, q < \infty$ .** First of all, we note that in the case when  $1 < q < \infty$  and  $f \in L_q(\mathbb{R}^d)$  the following relation holds (see, e.g., [21])

$$E_{Q(\mathcal{L})}(f)_q \leq \mathcal{E}_{Q(\mathcal{L})}(f)_q \leq C E_{Q(\mathcal{L})}(f)_q, \quad (10)$$

where  $C \geq 1$  be a positive constant.

The problem is to find the ordinal for parameter  $N$  estimations of quantities  $E_{Q(\mathcal{L})}(F)_q$  and  $\mathcal{E}_{Q(\mathcal{L})}(F)_q$  for  $F = S_{p,\theta}^\Omega B$  in the case when  $\mathcal{L} = \kappa(N)$  with certain limitations on the parameters  $p, q, \theta$  and  $\Omega$ .

**Theorem 1.** *Let  $1 < p < q < \infty$ ,  $1 \leq \theta \leq \infty$ ,  $\Omega \in \Phi_{\alpha,l}$  with  $\alpha > \frac{1}{p} - \frac{1}{q}$ . Then the following order estimates hold true:*

$$\mathcal{E}_{Q(N)}(S_{p,\theta}^\Omega B)_q \asymp E_{Q(N)}(S_{p,\theta}^\Omega B)_q \asymp \frac{1}{N} (\log_2 N)^{(d-1)(\frac{1}{q}-\frac{1}{\theta})_+}. \quad (11)$$

Note that the condition  $\Omega \in \Phi_{\alpha,l}$  with some  $\alpha > \beta = \frac{1}{p} - \frac{1}{q}$  ensures that for  $f \in S_{p,\theta}^\Omega B$  we have  $f \in S_{q,\theta}^{\Omega_1} B \subset L_q$ ,  $\Omega_1(\mathbf{t}) = \Omega(\mathbf{t})\mathbf{t}^{-\beta}$  and  $\|f\|_{S_{q,\theta}^{\Omega_1} B} \ll \|f\|_{S_{p,\theta}^\Omega B}$ .

**Proof.** First we prove the upper estimate in (11). Applying relation (10) and Lemma C for  $1 < q < \infty$  we obtain

$$\begin{aligned} E_{Q(N)}(f)_q &\asymp \mathcal{E}_{Q(N)}(f)_q = \left\| f(\cdot) - \sum_{\mathbf{s} \in \kappa(N)} \delta_{\mathbf{s}}^*(f, \cdot) \right\|_q = \left\| \sum_{\mathbf{s} \in \kappa^\perp(N)} \delta_{\mathbf{s}}^*(f, \cdot) \right\|_q \ll \\ &\ll \left( \sum_{\mathbf{s} \in \kappa^\perp(N)} \|\delta_{\mathbf{s}}^*(f, \cdot)\|_p^q 2^{\|\mathbf{s}\|_1(\frac{1}{p}-\frac{1}{q})q} \right)^{\frac{1}{q}}. \end{aligned} \quad (12)$$

Depending on the parameters  $q$  and  $\theta$ , we will consider few cases.

1) Let  $q < \theta$ . Then for  $1 < \theta < \infty$ , applying to (12) Hölder's inequality with the exponent  $\frac{\theta}{q}$  and the relations (8) and (9) we obtain

$$\begin{aligned} E_{Q(N)}(f)_q &\ll \left( \sum_{\mathbf{s} \in \kappa^\perp(N)} \|\delta_{\mathbf{s}}^*(f, \cdot)\|_p^\theta (\Omega(2^{-\mathbf{s}}))^{-\theta} \right)^{\frac{1}{\theta}} \times \\ &\times \left( \sum_{\mathbf{s} \in \kappa^\perp(N)} \left( 2^{\|\mathbf{s}\|_1(\frac{1}{p}-\frac{1}{q})} \Omega(2^{-\mathbf{s}}) \right)^{\frac{q\theta}{\theta-q}} \right)^{\frac{1}{q}-\frac{1}{\theta}} \leq \\ &\leq \|f\|_{S_{p,\theta}^\Omega B} \left( \sum_{\mathbf{s} \in \kappa^\perp(N)} \left( 2^{\|\mathbf{s}\|_1(\frac{1}{p}-\frac{1}{q})} \Omega(2^{-\mathbf{s}}) \right)^{\frac{q\theta}{\theta-q}} \right)^{\frac{1}{q}-\frac{1}{\theta}} \ll \frac{1}{N} |\Theta(N)|^{\frac{1}{q}-\frac{1}{\theta}} \asymp \\ &\asymp \frac{1}{N} (\log_2 N)^{(d-1)(\frac{1}{q}-\frac{1}{\theta})}. \end{aligned}$$

If  $\theta = \infty$  then for  $f \in S_{p,\infty}^\Omega B$ , according to the definition, a relationship  $\|\delta_{\mathbf{s}}^*(f, \cdot)\|_p \ll \Omega(2^{-\mathbf{s}})$  is true. From (9) we have

$$E_{Q(N)}(f)_q \ll \left( \sum_{\mathbf{s} \in \kappa^\perp(N)} \left( 2^{\|\mathbf{s}\|_1(\frac{1}{p}-\frac{1}{q})} \Omega(2^{-\mathbf{s}}) \right)^q \right)^{\frac{1}{q}} \ll \frac{1}{N} |\Theta(N)|^{\frac{1}{q}} \asymp$$



$$\asymp \frac{1}{N} (\log_2 N)^{\frac{d-1}{q}}.$$

2) Now let  $1 \leq \theta \leq q < \infty$ ,  $q \neq 1$ . Applying the inequality

$$\left( \sum_k |a_k|^{v_2} \right)^{\frac{1}{v_2}} \leq \left( \sum_k |a_k|^{v_1} \right)^{\frac{1}{v_1}}, \quad 0 < v_1 \leq v_2 < \infty,$$

(see., [35, p. 43]) and taking into account  $\Omega \in \Phi_{\alpha,l}$  with  $\alpha > \frac{1}{p} - \frac{1}{q}$  and (5) from (12) we obtain the estimate

$$\begin{aligned} E_{Q(N)}(f)_q &\ll \left( \sum_{\mathbf{s} \in \kappa^\perp(N)} \|\delta_{\mathbf{s}}^*(f, \cdot)\|_p^\theta (\Omega(2^{-\mathbf{s}}))^{-\theta} 2^{\|\mathbf{s}\|_1 (\frac{1}{p} - \frac{1}{q})\theta} (\Omega(2^{-\mathbf{s}}))^\theta \right)^{\frac{1}{\theta}} \ll \\ &\ll \left( \sum_{\mathbf{s} \in \kappa^\perp(N)} \|\delta_{\mathbf{s}}^*(f, \cdot)\|_p^\theta (\Omega(2^{-\mathbf{s}}))^{-\theta} \right)^{\frac{1}{\theta}} \sup_{\mathbf{s} \in \kappa^\perp(N)} 2^{\|\mathbf{s}\|_1 (\frac{1}{p} - \frac{1}{q})} \Omega(2^{-\mathbf{s}}) \leq \\ &\leq \|f\|_{S_{p,\theta}^\Omega B} \sup_{\mathbf{s} \in \kappa^\perp(N)} 2^{\|\mathbf{s}\|_1 (\frac{1}{p} - \frac{1}{q})} \Omega(2^{-\mathbf{s}}) \ll \frac{1}{N}. \end{aligned}$$

This proves the estimates from above in Theorem 1.

Now we will prove the estimates from below. To this end, for certain values on the parameters  $p$ ,  $q$  and  $\theta$  sufficient to indicate  $f \in S_{p,\theta}^\Omega B$  such that the lower estimates of  $\mathcal{E}_{Q(N)}(f)_q$  coincide with the order estimates from below of the quantities  $\mathcal{E}_{Q(N)}(S_{p,\theta}^\Omega B)_q$  in (11). First, we define the function on which construction will be carried out such functions  $f$ .

For  $\mathbf{x} = (x_1, \dots, x_d)$  we put

$$D_{\mathbf{k}}(\mathbf{x}) = \prod_{j=1}^d D_{k_j}(x_j), \quad \mathbf{k} \in \mathbb{Z}_+^d,$$

where

$$D_{k_j}(x_j) = \sqrt{\frac{2}{\pi}} \left( 2 \sin \frac{x_j}{2} \cos \frac{2k_j + 1}{2} x_j \right) \cdot x_j^{-1}.$$

It has been shown in [5] that for the Fourier transform of the function  $D_{\mathbf{k}}(\mathbf{x})$  we have

$$\mathfrak{F} D_{\mathbf{k}}(\mathbf{x}) = \chi_{\mathbf{k}}(\mathbf{x}) = \prod_{j=1}^d \chi_{k_j}(x_j),$$

where

$$\chi_{k_j}(x_j) = \begin{cases} 1, & \text{if } k_j < |x_j| < k_j + 1, \\ \frac{1}{2}, & \text{if } |x_j| = k_j \text{ or } |x_j| = k_j + 1, \\ 0 & \text{in all other cases,} \end{cases} \quad \chi_0(x_j) = \begin{cases} 1, & \text{if } |x_j| < 1, \\ \frac{1}{2}, & \text{if } |x_j| = 1, \\ 0, & \text{if } |x_j| > 1. \end{cases}$$

For the inverse Fourier transform we have

$$\mathfrak{F}^{-1}\chi_{\mathbf{k}}(\mathbf{t}) = D_{\mathbf{k}}(\mathbf{x}).$$

We also note that [5]

$$\left\| \sum_{\mathbf{k} \in \rho_+(\mathbf{s})} D_{\mathbf{k}}(\cdot) \right\|_p \asymp 2^{\|\mathbf{s}\|_1(1-\frac{1}{p})}, \quad (13)$$

where

$$\rho_+(\mathbf{s}) := \left\{ \mathbf{k} = (k_1, \dots, k_d) : \eta(s_j)2^{s_j-1} \leq k_j < 2^{s_j}, k_j \in \mathbb{Z}_+^d, j = \overline{1, d} \right\}.$$

We consider several cases depending on the parameters  $p, q$  and  $\theta$ .

Let  $\theta = \infty$ . We consider the function

$$f_1(\mathbf{x}) = C_7 \sum_{\mathbf{s} \in \Theta(N)} \Omega(2^{-\mathbf{s}}) 2^{-\|\mathbf{s}\|_1(1-\frac{1}{p})} \sum_{\mathbf{k} \in \rho_+(\mathbf{s})} D_{\mathbf{k}}(\mathbf{x}).$$

For a certain value of the constant  $C_7 > 0$  this function belongs to the class  $S_{p,\theta}^\Omega B$  because using the estimate (13), we can write

$$\begin{aligned} \|f_1(\cdot)\|_{S_{p,\infty}^\Omega B} &= \sup_{\mathbf{s} \in \Theta(N)} \frac{\|\delta_{\mathbf{s}}^*(f_1, \cdot)\|_p}{\Omega(2^{-\mathbf{s}})} = \\ &= \sup_{\mathbf{s} \in \Theta(N)} C_7 \frac{\left\| \Omega(2^{-\mathbf{s}}) 2^{-\frac{\|\mathbf{s}\|_1}{p}} \sum_{\mathbf{k} \in \rho_+(\mathbf{s})} D_{\mathbf{k}}(\cdot) \right\|_p}{\Omega(2^{-\mathbf{s}})} \leq C_8, \quad C_8 > 0. \end{aligned}$$

For  $\mathbf{s} \in \mathbb{Z}_+^d$  we put

$$\Delta(\mathbf{s}) = \left\{ \mathbf{x} : 2^{-s_j-1} \leq x_j < 2^{-s_j}, j = \overline{1, d} \right\},$$

and note that  $\Delta(\mathbf{s}) \cap \Delta(\mathbf{s}') = \emptyset$  if  $\mathbf{s} \neq \mathbf{s}'$ . Thus, taking into account that  $S_{Q(N)}(f_1, \cdot) = 0$ , using Theorem B and next relation (see, e.g., [5])

$$\begin{aligned} \left| \sum_{\mathbf{k} \in \rho(\mathbf{s})} D_{\mathbf{k}}(\mathbf{x}) \right| &= \left| \sum_{\mathbf{k} \in \rho(\mathbf{s})} \prod_{j=1}^d D_{k_j}(x_j) \right| = \left| \prod_{j=1}^d \sum_{k=\eta(s_j)2^{s_j-1}}^{2^{s_j}-1} D_{k_j}(x_j) \right| = \\ &= \left| \prod_{j=1}^d \sqrt{\frac{2}{\pi}} \frac{\sin 2^{s_j} x_j - \sin \eta(s_j) 2^{s_j-1} x_j}{x_j} \right|, \end{aligned}$$

we get

$$\begin{aligned} \mathcal{E}_{Q(N)}(S_{p,\theta}^\Omega B)_q &\geq \mathcal{E}_{Q(N)}(f_1)_q = \|f_1(\cdot)\|_q \gg \\ &\gg \left\| \left( \sum_{\mathbf{s} \in \Theta(N)} |\delta_{\mathbf{s}}^*(f_1, \cdot)|^2 \right)^{\frac{1}{2}} \right\|_q \geq \left( \sum_{\mathbf{s} \in \Theta(N)} \int_{\Delta(\mathbf{s})} |\delta_{\mathbf{s}}^*(f_1, \mathbf{x})|^q d\mathbf{x} \right)^{\frac{1}{q}} \gg \end{aligned}$$

$$\begin{aligned}
& \gg \left( \sum_{\mathbf{s} \in \Theta(N)} \int_{\Delta(\mathbf{s})} \left| \Omega(2^{-\mathbf{s}}) 2^{-\|\mathbf{s}\|_1(1-\frac{1}{p})} \sum_{\mathbf{k} \in \rho_+(\mathbf{s})} D_{\mathbf{k}}(\mathbf{x}) \right|^q d\mathbf{x} \right)^{\frac{1}{q}} = \\
& = \left( \sum_{\mathbf{s} \in \Theta(N)} \left( \Omega(2^{-\mathbf{s}}) 2^{-\|\mathbf{s}\|_1(1-\frac{1}{p})} \right)^q \int_{\Delta(\mathbf{s})} \left| \sum_{\mathbf{k} \in \rho_+(\mathbf{s})} D_{\mathbf{k}}(\mathbf{x}) \right|^q d\mathbf{x} \right)^{\frac{1}{q}} \gg \\
& \gg \left( \sum_{\mathbf{s} \in \Theta(N)} \left( \Omega(2^{-\mathbf{s}}) 2^{\|\mathbf{s}\|_1(\frac{1}{p}-\frac{1}{q})} \right)^q \right)^{\frac{1}{q}} \geq \frac{1}{2^l N} |\Theta(N)|^{\frac{1}{q}} \asymp \\
& \asymp \frac{1}{N} (\log_2 N)^{\frac{d-1}{q}}. \tag{14}
\end{aligned}$$

Now let  $1 \leq \theta \leq q < \infty$ ,  $q \neq 1$ . We consider the function

$$f_2(\mathbf{x}) := C_9 \Omega(2^{-\tilde{\mathbf{s}}}) 2^{-\|\tilde{\mathbf{s}}\|_1(1-\frac{1}{p})} \sum_{\mathbf{k} \in \rho_+(\tilde{\mathbf{s}})} D_{\mathbf{k}}(\mathbf{x}), \quad \tilde{\mathbf{s}} \in \Theta(N), \quad C_9 > 0.$$

According to (13) we have

$$\begin{aligned}
\|f_2(\cdot)\|_{S_{p,\theta}^\Omega B} & \asymp \left( \sum_{\mathbf{s} \in \Theta(N)} (\Omega(2^{-\mathbf{s}}))^{-\theta} \|\delta_{\mathbf{s}}^*(f_2, \cdot)\|_p^\theta \right)^{\frac{1}{\theta}} \ll \\
& \ll \left( (\Omega(2^{-\tilde{\mathbf{s}}}))^{-\theta} (\Omega(2^{-\tilde{\mathbf{s}}}))^\theta 2^{-\theta\|\tilde{\mathbf{s}}\|_1(1-\frac{1}{p})} \left\| \sum_{\mathbf{k} \in \rho_+(\tilde{\mathbf{s}})} D_{\mathbf{k}}(\cdot) \right\|_p^\theta \right)^{\frac{1}{\theta}} \asymp \\
& \asymp \left( 2^{-\theta\|\tilde{\mathbf{s}}\|_1(1-\frac{1}{p})} 2^{\theta\|\tilde{\mathbf{s}}\|_1(1-\frac{1}{p})} \right)^{\frac{1}{\theta}} = 1.
\end{aligned}$$

Therefore  $f_2 \in S_{p,\theta}^\Omega B$  with a certain value of the constant  $C_9$ .

Taking into account that  $S_{Q(N)}(f_2, \cdot) = 0$ , (13) and (5), we obtain

$$\begin{aligned}
\mathcal{E}_{Q(N)}(S_{p,\theta}^\Omega B)_q & \geq \mathcal{E}_{Q(N)}(f_2)_q = \|f_2(\cdot)\|_q \gg \Omega(2^{-\tilde{\mathbf{s}}}) 2^{-\|\tilde{\mathbf{s}}\|_1(1-\frac{1}{p})} \left\| \sum_{\mathbf{k} \in \rho_+(\tilde{\mathbf{s}})} D_{\mathbf{k}}(\cdot) \right\|_q \asymp \\
& \asymp \Omega(2^{-\tilde{\mathbf{s}}}) 2^{-\|\tilde{\mathbf{s}}\|_1(1-\frac{1}{p})} 2^{\|\tilde{\mathbf{s}}\|_1(1-\frac{1}{q})} = \Omega(2^{-\tilde{\mathbf{s}}}) 2^{\|\tilde{\mathbf{s}}\|_1(\frac{1}{p}-\frac{1}{q})} \geq \frac{1}{2^l N}.
\end{aligned}$$

In the cases  $1 < q < \theta < \infty$  for the function

$$f_3(\mathbf{x}) = C_{10} |\Theta(N)|^{-\frac{1}{\theta}} \sum_{\mathbf{s} \in \Theta(N)} \Omega(2^{-\mathbf{s}}) 2^{-\|\mathbf{s}\|_1(1-\frac{1}{p})} \sum_{\mathbf{k} \in \rho_+(\mathbf{s})} D_{\mathbf{k}}(\mathbf{x}),$$

using the relation (13), we obtain

$$\|f_3(\cdot)\|_{S_{p,\theta}^\Omega B} \asymp |\Theta(N)|^{-\frac{1}{\theta}} \left( \sum_{\mathbf{s} \in \Theta(N)} (\Omega(2^{-\mathbf{s}}))^{-\theta} \|\delta_{\mathbf{s}}^*(f_3, \cdot)\|_p^\theta \right)^{\frac{1}{\theta}} \asymp$$

$$\begin{aligned}
&\asymp |\Theta(N)|^{-\frac{1}{\theta}} \left( \sum_{\mathbf{s} \in \Theta(N)} (\Omega(2^{-\mathbf{s}}))^{-\theta} (\Omega(2^{-\mathbf{s}}))^{\theta} 2^{-\|\mathbf{s}\|_1 \theta (1-\frac{1}{p})} \left\| \sum_{\mathbf{k} \in \rho_+(\mathbf{s})} D_{\mathbf{k}}(\cdot) \right\|_p^{\theta} \right)^{\frac{1}{\theta}} \ll \\
&\ll |\Theta(N)|^{-\frac{1}{\theta}} \left( \sum_{\mathbf{s} \in \Theta(N)} 2^{-\|\mathbf{s}\|_1 \theta (1-\frac{1}{p})} 2^{\|\mathbf{s}\|_1 \theta (1-\frac{1}{p})} \right)^{\frac{1}{\theta}} = 1.
\end{aligned}$$

Thus  $f_3 \in S_{p,\theta}^{\Omega} B$  for a certain choice of the constant  $C_{10} > 0$ .

Taking into account that  $S_{Q(N)}(f_3, \cdot) = 0$  and making similar reasoning to (14), we obtain

$$\begin{aligned}
\mathcal{E}_{Q(N)}(S_{p,\theta}^{\Omega} B)_q &\geq \mathcal{E}_{Q(N)}(f_3)_q = \|f_3(\cdot)\|_q \gg \frac{1}{N} |\Theta(N)|^{\frac{1}{q}-\frac{1}{\theta}} \asymp \\
&\asymp \frac{1}{N} (\log_2 N)^{(d-1)(\frac{1}{q}-\frac{1}{\theta})}.
\end{aligned}$$

This completes the proof of the estimates from below in (11).

The Theorem 1 is proved.

Now we will establish the estimates of approximation of classes  $S_{p,\theta}^{\Omega} B(\mathbb{R}^d)$  in the space  $L_q(\mathbb{R}^d)$ ,  $1 < q < p < \infty$ .

First we formulate known statement.

**Theorem C [1].** *Let  $1 < p < \infty$ ,  $1 \leq \theta \leq \infty$ ,  $\Omega \in \Phi_{\alpha,l}$ . Then the following order estimates hold true:*

$$\mathcal{E}_{Q(N)}(S_{p,\theta}^{\Omega} B)_p \asymp E_{Q(N)}(S_{p,\theta}^{\Omega} B)_p \asymp \frac{1}{N} (\log_2 N)^{(d-1)(\frac{1}{p_0}-\frac{1}{\theta})_+}, \quad (15)$$

where  $p_0 := \min\{p, 2\}$ .

Further, we put

$$S_{p,q,\theta}^{\Omega} B(\mathbb{R}^d) := S_{p,\theta}^{\Omega} B(\mathbb{R}^d) \cap L_q(\mathbb{R}^d).$$

**Theorem 2.** *Let  $1 < q < p < \infty$ ,  $p \geq 2$ ,  $1 \leq \theta \leq \infty$ ,  $\Omega \in \Phi_{\alpha,l}$ ,  $\alpha > 0$ . Then the following order estimates hold true:*

$$\mathcal{E}_{Q(N)}(S_{p,q,\theta}^{\Omega} B)_q \asymp E_{Q(N)}(S_{p,q,\theta}^{\Omega} B)_q \asymp \frac{1}{N} (\log_2 N)^{(d-1)(\frac{1}{2}-\frac{1}{\theta})_+}. \quad (16)$$

**Proof.** Estimates from above in (16) are a consequence of (15).

Now we will prove the estimates from below. We consider several cases depending on the parameter  $\theta$ . Also we use the following statement [5].

**Lemma D.** *Let  $1 < p < \infty$ . Then for the function*

$$f(\mathbf{x}) = \sum_{\mathbf{k} \geq 0} c_{\mathbf{k}} \prod_{j=1}^d D_{2^{k_j-1}}(x_j)$$

we have

$$\|f\|_p \asymp \left( \sum_{\mathbf{k} \geq 0} |c_{\mathbf{k}}|^2 \right)^{\frac{1}{2}}.$$

In the case when  $2 \leq \theta < \infty$ , we consider the function

$$\psi_1(\mathbf{x}) = C_{11} \frac{1}{N} |\Theta(N)|^{-\frac{1}{\theta}} \sum_{\mathbf{s} \in \Theta(N)} \prod_{j=1}^d D_{2^{s_j-1}}(x_j).$$

We show that  $\psi_1 \in S_{p,q,\theta}^\Omega B$  with some constant  $C_{11} > 0$ .

Indeed, since

$$\left\| \prod_{j=1}^d D_{2^{s_j-1}}(x_j) \right\|_p \ll \left\| \prod_{j=1}^d \left| \frac{\sin(x_j/2)}{x_j} \right| \right\|_p \ll 1, \quad (17)$$

is considering the relation (5), we get

$$\begin{aligned} \|\psi_1\|_{S_{p,\theta}^\Omega B} &\asymp \left( \sum_{\mathbf{s} \in \Theta(N)} (\Omega(2^{-\mathbf{s}}))^{-\theta} \left\| \delta_{\mathbf{s}}^*(\psi_1, \cdot) \right\|_p^\theta \right)^{\frac{1}{\theta}} = \\ &= C_{11} \frac{1}{N} |\Theta(N)|^{-\frac{1}{\theta}} \left( \sum_{\mathbf{s} \in \Theta(N)} (\Omega(2^{-\mathbf{s}}))^{-\theta} \left\| \prod_{j=1}^d D_{2^{s_j-1}}(\cdot) \right\|_p^\theta \right)^{\frac{1}{\theta}} \ll \\ &\ll \frac{1}{N} |\Theta(N)|^{-\frac{1}{\theta}} \left( \sum_{\mathbf{s} \in \Theta(N)} (\Omega(2^{-\mathbf{s}}))^{-\theta} \right)^{\frac{1}{\theta}} \ll \\ &\ll \frac{1}{N} |\Theta(N)|^{-\frac{1}{\theta}} \left( \frac{1}{N} \right)^{-1} \left( \sum_{\mathbf{s} \in \Theta(N)} 1 \right)^{\frac{1}{\theta}} = 1. \end{aligned}$$

Hence, if the constant  $C_{11}$  is chosen properly, then  $\psi_1 \in S_{p,\theta}^\Omega B$ .

On the other side, using Lemma D, we obtain

$$\begin{aligned} \|\psi_1(\cdot)\|_q &\asymp \frac{1}{N} |\Theta(N)|^{-\frac{1}{\theta}} \left\| \sum_{\mathbf{s} \in \Theta(N)} \prod_{j=1}^d D_{2^{s_j-1}}(x_j) \right\|_q \asymp \\ &\asymp \frac{1}{N} |\Theta(N)|^{-\frac{1}{\theta}} \left( \sum_{\mathbf{s} \in \Theta(N)} 1 \right)^{\frac{1}{2}} \asymp \frac{1}{N} |\Theta(N)|^{\frac{1}{2}-\frac{1}{\theta}} \asymp \frac{1}{N} (\log_2 N)^{(d-1)(\frac{1}{2}-\frac{1}{\theta})}. \end{aligned} \quad (18)$$

Therefore  $\psi_1 \in L_q(\mathbb{R}^d)$ ,  $1 < q < \infty$ .

Further, taking into account that  $S_{Q(N)}(\psi_1, \cdot) = 0$  and (18), we get

$$\mathcal{E}_{Q(N)}(S_{p,q,\theta}^\Omega B)_q \geq \mathcal{E}_{Q(N)}(\psi_1)_q = \|\psi_1(\cdot)\|_q \asymp \frac{1}{N} (\log_2 N)^{(d-1)(\frac{1}{2}-\frac{1}{\theta})}.$$

If  $\theta = \infty$ , we consider the function

$$\psi_2(\mathbf{x}) = C_{12} \frac{1}{N} \sum_{\mathbf{s} \in \Theta(N)} \prod_{j=1}^d D_{2^{s_j-1}}(x_j),$$

where  $C_{12} > 0$  be a some constant.

By (2), (5) and (17), we get

$$\|\psi_2\|_{S_{p,\infty}^\Omega B} \asymp \sup_{s \in \Theta(N)} \frac{\|\delta_s^*(\psi_2, \cdot)\|_p}{\Omega(2^{-s})} = \frac{C_{12}}{N} \sup_{s \in \Theta(N)} \frac{\left\| \prod_{j=1}^d D_{2^{s_j-1}}(x_j) \right\|_p}{\Omega(2^{-s})} \leq C_{13},$$

So  $\psi_2 \in S_{p,\infty}^\Omega B$  for a certain value of the constant  $C_{12} > 0$ .

For the norm of function  $\psi_2$  in space  $L_q(\mathbb{R}^d)$  by Lemma D, we have

$$\begin{aligned} \|\psi_2(\cdot)\|_q &\asymp \frac{1}{N} \left\| \sum_{s \in \Theta(N)} \prod_{j=1}^d D_{2^{s_j-1}}(x_j) \right\|_q \asymp \\ &\asymp \frac{1}{N} \left( \sum_{s \in \Theta(N)} 1 \right)^{\frac{1}{2}} = \frac{1}{N} |\Theta(N)|^{\frac{1}{2}} \asymp \frac{1}{N} (\log_2 N)^{\frac{d-1}{2}}. \end{aligned} \quad (19)$$

Thus  $\psi_2 \in S_{p,\infty}^\Omega(\mathbb{R}^d) \cap L_q(\mathbb{R}^d)$ .

Taking into account that  $S_{Q(N)}(\psi_2, \cdot) = 0$  and (19) we have

$$\mathcal{E}_{Q(N)}(S_{p,q,\infty}^\Omega B)_q \geq \mathcal{E}_{Q(N)}(\psi_2)_q = \|\psi_2(\cdot)\|_q \asymp \frac{1}{N} (\log_2 N)^{\frac{d-1}{2}}.$$

To establish the lower estimates of quantities  $E_{Q(N)}(S_{p,q,\theta}^\Omega B)_q$  in the case  $1 \leq \theta < 2$  we consider the function

$$\psi_3(\mathbf{x}) = C_{14} \Omega(2^{-\tilde{s}}) \prod_{j=1}^d D_{2^{\tilde{s}_j-1}}(x_j),$$

where  $\tilde{s} \in \Theta(N)$ ,  $C_{14} > 0$  be a some constant.

Using (17) and (5) to the function  $\psi_3$ , we obtain

$$\|\psi_3(\cdot)\|_q \asymp \Omega(2^{-\tilde{s}}) \asymp \frac{1}{N} \quad (20)$$

and

$$\begin{aligned} \|\psi_3(\cdot)\|_{S_{p,\theta}^\Omega} &\asymp \left( (\Omega(2^{-\tilde{s}}))^{-\theta} \left\| \Omega(2^{-\tilde{s}}) \prod_{j=1}^d D_{2^{\tilde{s}_j-1}}(x_j) \right\|_p^\theta \right)^{\frac{1}{\theta}} = \\ &= \left( \left\| \prod_{j=1}^d D_{2^{\tilde{s}_j-1}}(x_j) \right\|_p^\theta \right)^{\frac{1}{\theta}} \leq C_{15}. \end{aligned}$$

Thus  $\psi_3 \in S_{p,q,\theta}^\Omega B$  for a certain value of the constant  $C_{14} > 0$ .

Since  $S_{Q(N)}(\psi_3, \cdot) = 0$  then using (20), we obtain

$$\mathcal{E}_{Q(N)}(S_{p,q,\theta}^\Omega B)_q \geq \mathcal{E}_{Q(N)}(\psi_3)_q = \|\psi_3(\cdot)\|_q \asymp \frac{1}{N}.$$

The lower bounds in (16) are established.

Theorem 2 is proved.

**Theorem 3.** *Let  $1 < q < p < 2$ ,  $1 \leq \theta \leq p$ ,  $\Omega \in \Phi_{\alpha,l}$ ,  $\alpha > 0$ . Then the following order estimate is true:*

$$\mathcal{E}_{Q(N)}(S_{p,q,\theta}^\Omega B)_q \asymp E_{Q(N)}(S_{p,q,\theta}^\Omega B)_q \asymp \frac{1}{N}. \quad (21)$$

**Proof.** Estimate from above in (21) is a consequence of (15). To establish the lower estimates is sufficient to consider the function  $\psi_3$  from Theorem 2.

### 3. Comments.

Now we make a comment about of our results which was obtained in Theorem 1 and Theorem 2.

Let  $\omega(\tau)$  be a function of one variable  $\omega \in \Phi_{\alpha,l}$ ,  $\alpha > 0$ , and the function of mixed modulus of smoothness of order  $l$  is given by

$$\Omega(\mathbf{t}) = \Omega(t_1, \dots, t_d) = \omega\left(\prod_{j=1}^d t_j\right).$$

For this functional parameter  $\Omega$  the order estimation of quantity  $E_{\bar{Q}_n}(S_{p,\theta}^\Omega B)_q$ , in the case  $1 < p < q < \infty$ ,  $\alpha > \frac{1}{p} - \frac{1}{q}$ , where  $\bar{Q}_n = \bigcup_{\|\mathbf{s}\|_1 < n} Q^*(\mathbf{s})$ , have been obtained in [36] and

particularly in the case  $\Omega(\mathbf{t}) = \prod_{j=1}^d t_j^r$ ,  $\frac{1}{p} - \frac{1}{q} < r < l$ , have been obtained in [5]. Note that in [5] also considered the case when  $\mathbf{r} = (r_1, \dots, r_d)$ ,  $r_j > \frac{1}{p} - \frac{1}{q}$ ,  $j = \overline{1, d}$ .

If  $1 < q < p < \infty$ ,  $p \geq 2$ ,  $\alpha > 0$ , then the order estimation of quantity  $E_{\bar{Q}_n}(S_{p,\theta}^\Omega B)_q$  have been obtained in [36].

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